The Scientific Journal of King Faisal University

Basic and Applied Sciences

The Tensor Product of Zero-Divisor Graphs of Variation Monogenic Semigroups

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LINK	RECEIVED	ACCEPTED	PUBLISHED ONLINE	ASSIGNED TO AN ISSUE
https://doi.org/10.37575/b/sci/210038	31/05/2021	27/08/2021	27/08/2021	01/12/2021

NO. OF WORDS	NO. OF PAGES	YEAR	VOLUME	ISSUE
3381	4	2021	22	2

ABSTRACT

The tensor product of zero-divisor graphs of variation monogenic semigroups $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$ is studied. The vertices(x_1^i, x_2^j) and (x_1^k, x_2^f) of the tensor product of this graph are adjacent whenever $gcd(i,k) = 1, i + k > n, gcd(j,f) = 1 , j + f > m$. Some properties of tensor product graphs are obtained, such as girth, diameter, chromatic, clique and domination numbers.

KEYWORDS

Variation monogenic semigroup, relatively prime, tensor product, adjacency

CITATION

Akwu, A.D. and Al Subaiei, B. (2021). The tensor product of zero-divisor graphs of variation monogenic semigroups. *The Scientific Journal of King Faisal University: Basic and Applied Sciences*, **22**(2), 61–4. DOI: 10.37575/b/sci/210038

1. Introduction

The zero-divisor graphs on semigroups were widely investigated in many studies; for example, DeMeyer *et al.* (2002), DeMeyer *et al.* (2005), Wright (2007) and Anderson and Badawi (2017).

Das *et al.* (2013) studied zero-divisor graphs of the monogenic semigroup $\Gamma(S_M)$, which is defined as $x^i.x^j = x^{i+j} = 0$ if and only if $i + j > n$. Then, Al Subaiei and Akwu (2020) modified the graph in the work of Das *et al.* (2013) by adding one more condition, $gcd(i,j) = 1$, and named the graph the variation monogenic semigroup $\Gamma(VS_M)$. These two graphs have different properties. For example, when $n = 8$, the graph of Das *et al.* (2013) has $diam(\Gamma(S_{M8})) = 2$, $\mathcal{X}(\Gamma(S_{M8}) = 5$, $\gamma(\Gamma(S_{M8})) = 1$, and $\omega(\Gamma(S_{M8})) = 5$. Meanwhile, the graph of Al Subaiei and Akwu (2020) has $diam(\Gamma(VS_{M8})) = 3$, $\mathcal{X}(\Gamma(VS_{M8}) = 3$, $\gamma(\Gamma(VS_{M8})) = 2$, and $\omega(\Gamma(VS_{M8})) = 3$. Also, when n is prime, the two graphs have different properties. For example, when $n = 7$, the graph of Das *et al.* (2013) has $\mathcal{X}(\Gamma(S_{M7}) = 4, \gamma(\Gamma(S_{M7})) = 1$, and $\omega(\Gamma(S_{M7})) = 4$, while the graph of Al Subaiei and Akwu (2020) has $\mathcal{X}(\Gamma(VS_{M7}) = 3, \gamma(\Gamma(VS_{M7})) = 1$, and $\omega(\Gamma(VS_{M7})) = 3$. These differences led us to further investigate the results of the properties of the variation monogenic semigroup graph, such as the tensor product.

The concept of the tensor product on graphs has been studied well and a sufficient amount of rich material can be found in the literature. Some examples are Harary and Trauth, Jr. (1966), Sampathkumar (1972) and Asmerom (1998). The tensor product of zero-divisor graphs of monogenic semigroups was investigated by Akgunes *et al.* (2014). This work will extend the investigation of the monogenic semigroup graph of Akgunes *et al.* (2014) to the variation monogenic semigroup graph and hence will use the same notation.

A set with an associative binary operation is called a semigroup. The semigroup that is generated by one element is called a monogenic semigroup. A monogenic semigroup with zero is denoted by S_{Mn} where n is the order of the semigroup. The nonzero elements of S_{Mn} are referred to as the vertices of the graph $\Gamma(S_{Mn})$. In Al Subaiei and Akwu's work (2020), the undirected graph of the variation monogenic semigroup with order n denoted by $\Gamma(VS_{Mn})$ was given, and the two vertices x^i and x^j are adjacent if and only if the following conditions are satisfied:

$x^i.x^j = x^{i+j} = 0$ if and only if $i + j > n$ and $gcd(i,j) = 1 (*)$

where x^i and $x^j \in V(\Gamma(VS_{Mn}))$ and $1 \le i,j \le n$.

For basic general information about graph theory properties, we refer the reader to Bondy and Murty (1976). In general, for any graph G, $V(G)$ is known as the set of vertices in G. If the two vertices x, y are adjacent, then $xy \in E(G)$. The distance (shortest path) between two vertices is denoted by $d_{\Gamma(G)}$. The *diameter*, denoted by $diam(\Gamma(G))$ is defined as $diam(\Gamma(G)) = max\{d_{\Gamma(G)}(x,y): x, y \in V(\Gamma(G))\}$. The length of the shortest cycle in the graph is known as the *girth*. The *degree* of the vertex x^i, denoted by $deg_{\Gamma(G)}(x^i)$, is the number of vertices that are adjacent to x^i. The maximum degree of $\Gamma(G)$ is denoted by $\Delta(\Gamma(G))$, while the minimum degree of $\Gamma(G)$ is denoted by $\delta(\Gamma(G))$, which means the number of the largest and smallest vertex's degrees, respectively. The subset D of $V(G)$ is a *dominating set* of G if each vertex of $V(G)\backslash D$ is adjacent to at least one vertex of D. The *domination number*, $\gamma(G)$, is the dominating set with minimum cardinality. The *clique number*, $\omega(G)$, is the maximum number of vertices in any clique where the clique is a complete subgraph of graph G. The *chromatic number* of G, $\mathcal{X}$ is the minimum number of colors assigned to the vertices of G such that no two adjacent vertices have the same color. When $\omega(G) = \mathcal{X}(G)$, the graph G is a *perfect graph,* according to Lovasz (1972). For a graph G with order n, a *coprime labeling* of G is a labeling of its vertices with distinct integers $\{1,2, \dots, n\}$ such that the labels on any two adjacent vertices are relatively prime. A *coprime graph* is a graph that has coprime labeling.

Consider the simple graphs G_1 and G_2. It is known that the *tensor product* $G_1 \otimes G_2$ has the vertex set $V(G_1) \times V(G_2)$, where any two vertices, (g_1, g_2) and (h_1, h_2), are adjacent if and only if $g_1 h_1 \in E(G_1)$ and $g_2 h_2 \in E(G_2)$.

In our work, we will consider the graphs of the variation monogenic semigroup $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$ and study the tensor product of these graphs. It is known in the theory of graphs that when any two vertices, (x_1^i, x_2^j) and (x_1^k, x_2^f) are adjacent, it means that $(x_1^i, x_2^j)(x_1^k, x_2^f) \in E(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2))$. In this work, two adjacent vertices, (x_1^i, x_2^j) and (x_1^k, x_2^f) , will be written as $(x_1^i, x_2^j) - (x_1^k, x_2^f)$. Since the graphs $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$ are

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simple, then (x_1^i, x_2^j) is not adjacent to (x_1^k, x_2^f) whenever $i = k$ or $j = f$. Therefore, $(x_1^i, x_2^j) - (x_1^k, x_2^f)$ whenever $x_1^i - x_1^k$ and $x_2^j - x_2^f$. That is:

$$x_1^i x_1^k \in E\Big(\Gamma(VS_{Mn}^1)\Big) \,\&\, x_2^j x_2^f \in E(\Gamma(VS_{Mm}^2))$$
$$\Updownarrow$$
$$gcd(i,k) = 1, i + k > n, gcd(j,f) = 1\,, j + f > m$$

2. Results

In this section, we give some tensor product characteristics of the variation monogenic semigroup graphs $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$. The condition of the greatest common divisor on the graph of the variation monogenic semigroup makes the structure of the proofs in this work depend on the highest prime numbers p_1 and p_2 being less than or equal to n or m, respectively.

The following results give the maximum degree and minimum degree of the tensor product graph $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$.

2.1. Theorem: Let p_1 and p_2 be the highest prime numbers less or equal to n and m, respectively. Then, the maximum and minimum degrees of $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$ *are* $\Delta((\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2))) = (p_1 - 1)(p_2 - 1)$ and $\delta((\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2))) = 1$ respectively.

Proof. When $p_1 = n$ and $p_2 = m$, then $gcd(p_1, j) = 1$ and $gcd(p_1, k) = 1$ where $1 \le j < n$ and $1 \le k < m$. Hence, the result follows from Akgunes *et al.* (2014). Otherwise, we assume that $p_1 \ne n$, and $p_2 \ne m$, then vertex $(x_1^{p_1}, x_2^{p_2})$ has the maximum degree. Therefore, $(x_1^{p1}, x_2^{p_2}) - (x_1^d, x_2^s)$ when $p_1 + d > n$, $gcd(p_1, d) = 1$, $p_2 + s > m$ and $gcd(p_2, s) = 1$. Hence, $d > n - p_1$ and $s > m - p_2$. Since the graph is a simple graph, $(x_1^{p_1}, x_2^{p_2})$ is not adjacent to itself. Therefore, $deg((x_1^{p_1}, x_2^{p_2})) = |d||s|$, where $|d| = (n - (n - p_1)) - 1 = p_1 - 1$ and $|s| = (m - (m - p_2)) - 1 = p_2 - 1$. Thus, the maximum degree is $(p_1 - 1)(p_2 - 1)$.

Since 1 is relatively prime with any number and $n + 1 > n$ and $m + 1 > m$, the vertex (x_1, x_2) is adjacent only to (x_1^n, x_2^m). So, the minimum degree is 1.

The following two results are on the diameter and girth of the tensor product graph $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$. The diameter of the tensor product graphs of the monogenic semigroup $\Gamma(S_{Mn}^1) \otimes \Gamma(S_{Mm}^2)$, as shown in Akgunes *et al.* (2014), is 4 when $n \ge m > 3$, and the girth is 3 when $n \ge m > 3$. However, in the case of the tensor product graphs of the variation monogenic semigroup, the diameter is 4 when $n \ge m > 6$, and the girth is 3 when $n, m > 4$ as shown below.

2.2. Theorem: Let $n \ge m$ be positive integers. For any variation monogenic semigroup graphs $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$ with order n and m, respectively, $diam\Big(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)\Big) =$

$$\begin{cases} 4, & if\ m > 6 \\ 5, & if\ m \le 6, n > 4\ and\ (n,m) \ne (6,m), where\ gcd(6,m) \ne 1 \\ 7, & if\ (n,m) = (6,2), (6,3), (6,4) \end{cases}$$

Proof. We split the problem into the following cases:

Case 1: If $m > 6$, the diameter can be considered as the distance between (x_1, x_2) and (x_1, x_2^2) whenever n is a positive integer and m is even, which is given below:

$$(x_1, x_2) - (x_1^n, x_2^m) - (x_1^p, x_2^p) - (x_1^n, x_2^{m-1}) - (x_1, x_2^2)$$

where p is a prime number less than n and m in the first and second coordinate, respectively. Also, fr n a positive integer and m odd, the diameter can be viewed as the distance between (x_1, x_2) and (x_1^n, x_2) given as follows:

$$(x_1, x_2) - (x_1^n, x_2^m) - (x_1^p, x_2^p) - (x_1^{n-1}, x_2^m) - (x_1^n, x_2)$$

Hence, the diameter is 4.

Case 2: If $m \le 6, n > 4$ and $(n,m) \ne (6,3), (6,4)$. The diameter of the graph can be viewed as the distance between (x_1, x_2) and (x_1, x_2^m) whenever m is even and is given below:

$$(x_1, x_2) - (x_1^n, x_2^m) - (x_1^p, x_2^p) - (x_1^{p-2}, x_2^m) - (x_1^n, x_2) - (x_1, x_2^m)$$

Also, whenever m is odd, the diameter can be viewed as the distance between (x_1, x_2) and (x_1, x_2^m), given as:

$$(x_1, x_2) - (x_1^n, x_2^m) - (x_1^p, x_2^p) - (x_1^{n-1}, x_2^m) - (x_1^n, x_2) - (x_1, x_2^m)$$

Hence, the diameter is 5.

Case 3: Whenever $(n,m) = (6,2)$, the diameter can be viewed as the distance between (x_1, x_2) and (x_1, x_2^2), given as:

$$(x_1, x_2) - (x_1^6, x_2^2) - (x_1^5, x_2^1) - (x_1^3, x_2^2) - (x_1^4, x_2^1) - (x_1^5, x_2^2) - (x_1^6, x_2^1) - (x_1^1, x_2^2).$$

Whenever $(n,m) = (6,3)$, the diameter can be viewed as the distance between (x_1, x_2) and (x_1, x_2^3), given as:

$$(x_1, x_2) - (x_1^6, x_2^3) - (x_1^5, x_2^2) - (x_1^2, x_2^3) - (x_1^4, x_2^2) - (x_1^6, x_2^3) - (x_1^6, x_2^1) - (x_1^1, x_2^3).$$

Whenever $(n,m) = (6,4)$, the diameter can be viewed as the distance between (x_1, x_2) and (x_1, x_2^2), given as:

$$(x_1, x_2) - (x_1^6, x_2^4) - (x_1^5, x_2^3) - (x_1^3, x_2^2) - (x_1^4, x_2^3) - (x_1^5, x_2^2) - (x_1^6, x_2^3) - (x_1^1, x_2^2)$$

Hence, the diameter is 7.

2.3. Remark: Whenever $(n,m) = (6,6)$, the diameter is 6, which can be viewed as the distance between (x_1, x_2) *and* (x_1, x_2^2), given as:

$$(x_1, x_2) - (x_1^6, x_2^6) - (x_1^1, x_2^5) - (x_1^6, x_2^4) - (x_1^1, x_2^3) - (x_1^6, x_2^5) - (x_1^6, x_2^2).$$

2.4. Theorem: For any variation monogenic semigroup graphs $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$, the girth of the graph $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$ is

$$= \begin{cases} 3, & if\ n, m > 4 \\ 4, & if\ n > 4, and\ 1 < m \le 4 \\ 4, & if\ 1 < n \le 4, and\ m > 4 \\ not\ exists & if\ n \le 4\ and\ m \le 4 \end{cases}$$

Proof. Let p_1 and p_2 be the highest prime numbers less or equal to n and m, respectively. By definition of $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$, for $n, m > 4$ we have:

$$(x_1^{p_1}, x_2^{p_2}) - (x_1^{p_1-1}, x_2^{p_2-1}) - (x_1^{p_1-2}, x_2^{p_2-2}) - (x_1^{p_1}, x_2^{p_2})$$

Since when $n, m > 4$ we have $gcd(p_i, p_i - 1) = 1$, $p_i + p_i - 1 > n, m$, $gcd(p_i - 1, p_i - 2) = 1$, $p_i - 1 + p_i - 2 > n, m$, $gcd(p_i, p_i - 2) = 1$, and $p_i + p_i - 2 > n, m$. So, $x_i^{p_i} - x_i^{p_i - 1}$, $x_i^{p_i-1} - x_i^{p_i-2}$, and $x_i^{p_i} - x_i^{p_i-2}$, where $i = 1$ or 2.

For $n > 4$ and $1 < m \le 4$, we have that, if $m = 4$ or 3:

$$(x_1^{p_1}, x_2^3) - (x_1^{p_1-1}, x_2^2) - (x_1^{p_1-2}, x_2^3) - (x_1^{p_1-1}, x_2^2) - (x_1^{p_1}, x_2^3)$$

while if $m = 2$:

$$(x_1^{p_1}, x_2^2) - (x_1^{p_1-1}, x_2^1) - (x_1^{p_1-2}, x_2^2) - (x_1^{p_1-1}, x_2^1) - (x_1^{p_1}, x_2^2)$$

For $1 < n \le 4$ and $m > 4$, by using a similar argument as above, the result follows (exchange m with n in the above case).

For $n, m \le 4$, it is clear that $x_i^{p-1} x_i^{p-2} \ne 0$ since $2p - 3 \ngtr n, m$. Therefore, $(x_1^{p_1-1}, x_2^{p_2-1})$ is not adjacent to $(x_1^{p_1-2}, x_2^{p_2-2})$, which implies that there is no cycle connecting $(x_1^{p_1}, x_2^{p_2})$ for $n, m \le 4$.

Hence, the girth does not exist in this case.

For $n = 1$, it is clear from the definition of the tensor product that there is no vertex adjacent to the vertex (x_1, x_2^k). Hence, the result follows.

2.5. Example: Consider the graph $\Gamma(VS_{M5}) \otimes \Gamma(VS_{M10})$. Then, from the above results, $\Delta(\Gamma(VS_{M5}) \otimes \Gamma(VS_{M10})) = 24$, $\delta(\Gamma(VS_{M5}) \otimes \Gamma(VS_{M10})) = 1$, $diam(\Gamma(VS_{M5}) \otimes \Gamma(VS_{M10})) = 4$, and $girth(\Gamma(VS_{M5}) \otimes \Gamma(VS_{M10})) = 3$. While, the graph $\Gamma(VS_{M7}) \otimes \Gamma(VS_{M4})$ *has* $\Delta(\Gamma(VS_{M7}) \otimes \Gamma(VS_{M4})) = 12$, $\delta(\Gamma(VS_{M7}) \otimes \Gamma(VS_{M4})) = 1$, $diam(\Gamma(VS_{M7}) \otimes \Gamma(VS_{M4})) = 5$, and $girth(\Gamma(VS_{M7}) \otimes \Gamma(VS_{M4})) = 4$.

The following definition apeared in Al Subaiei and Akwu (2020).

2.6. Definition: For a postive integer n, the set $\pi(n) = \{p_1, p_2, \ldots, p_k\}$ is the set of consecutive prime numbers less or equal to n such that $p_t + p_{t+1} > n$ and $\pi^*(n) = |\pi(n)|$. Also, for positive integers s_f^* where $\lceil \frac{n}{2} \rceil < s_f^* \leq n$ and the following is satisfied:

1. $s_f^* \notin \pi(n)$
2. $gcd(s_f^*, p_f) = 1$ for all $p_f \in \pi(n)$
3. $gcd(s_f^*, s_l^*) = 1$ for any pair $\lceil \frac{n}{2} \rceil < s_f^*, s_l^* \leq n$

the numbers of s_f^* are denoted by $S^* = |s_f^*|$.

For simplicity, suppose that the graph $\Gamma(VS_{Mn}^1)$ has $\pi^*(n)$ and S_1^*, while the graph $\Gamma(VS_{Mm}^2)$ has $\pi^*(m)$ and S_2^*. It is also known from Al Subaiei and Akwu (2020) that $\mathcal{X}(\Gamma(VS_{Mn})) = \omega(\Gamma(VS_{Mn})) = \pi^*(n) + S_1^*$. By using these facts, we establish the following theorems on the chromatic number, clique number and domination number of $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$.

2.7. Theorem: For any variation monogenic semigroup graphs $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$, the chromatic number of $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$ is given as:

$$\mathcal{X}(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)) = \begin{cases} \pi^*(n) + S_1^*, & if\ n \leq m \\ \pi^*(m) + S_2^*, & if\ m < n \end{cases}$$

Proof. Suppose that $n < m$. From the definition of $\pi(n)$ and $\pi(m)$, we know that $(x_1^{p_f}, x_2^{p_l}) - (x_1^{p_t} x^{p_h})$ for any $p_f, p_t \in \pi(n)$ and $p_l, p_h \in \pi(m)$. Also, we know from the definition of the tensor product that $(x_1^{p_f}, x_2^{p_l})$ is neither adjacent to $(x_1^{p_t}, x_2^{p_l})$ nor $(x_1^{p_f}, x^{p_h})$. Since $n < m$, $\pi^*(n) \leq \pi^*(m)$. Then, we can assign all vertices that have the same first coordinator to the same color. For example, all $p_l \in \pi(m)$, $(x_1^{p_f}, x_2^{p_l})$ will be in the same color. Therefore, we need $\pi^*(n)$ distinct colors.

Consider the vertex $(x_1^{p_f}, x_2^{p_l})$ where $p_f \in \pi(n)$ and $p_l \in \pi(m)$. Then, for any prime numbers r, z that satisfy $r \notin \pi(n)$ or $z \notin \pi(m)$, it is clear that the vertices (x_1^r, x_2^j) and (x_1^i, x_2^z) where $1 \leq i \leq n$ and $1 \leq j \leq m$ can be assigned the same color as $(x_1^{p_f}, x_2^{p_l})$. If color C_1 is assigned to vertex $(x_1^{p_f}, x_2^{p_l})$ where $p_f \in \pi(n)$ and $p_l \in \pi(m)$, then it is obvious that $x_1^r x_1^{p_f} \notin E(\Gamma(VS_{Mn}^1))$ and $x_2^z x_2^{p_l} \notin E(\Gamma(VS_{Mn}^2))$. This implies that color C_1 can be assigned to all vertices (x_1^r, x_2^j) and (x_1^i, x_2^z) where $1 \leq i \leq n$ and $1 \leq j \leq m$.

Now, consider the set of vertices $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*}), (x_1^{s_{1e}^*}, x_2^{s_{2q}^*})$ such that $s_{1f}^*, s_{1e}^* \notin \pi(n)$, $s_{2l}^*, s_{2q}^* \notin \pi(m)$, $\lceil \frac{n}{2} \rceil < s_{1f}^*, s_{1e}^* \leq n$, and $\lceil \frac{m}{2} \rceil < s_{2l}^*, s_{2q}^* \leq m$. From the definition of s_{1f}^* and s_{2l}^*, we have $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*}) - (x_1^{s_{1e}^*}, x_2^{s_{2q}^*})$ and $(x_1^{p_f}, x_2^{p_l}) - (x_1^{s_{1f}^*}, x_2^{s_{2l}^*})$. Moreover, we know from the definition of the tensor product that $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*})$ is neither adjacent with $(x_1^{s_{1f}^*}, x_2^{s_{2q}^*})$ nor $(x_1^{s_{1e}^*}, x_2^{s_{2l}^*})$. Since $n < m$, $S_1^* \leq S_2^*$. Therefore, S_1^* needs more colors for the vertices $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*})$. The leftover vertices (x_1^g, x_2^h), where $1 < g \leq n$ and $1 < h \leq m$, can be assigned to one of the colors assigned to vertices $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*})$ if $gcd(g, s_{1f}^*) \neq 1$ and $gcd(h, s_{2l}^*) \neq 1$. The vertex (x_1, x_2) can be assigned to any color that has not been assigned to (x_1^n, x_2^m). Therefore, there are $\pi^*(n) + S_1^*$ colors needed to color the graph $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$. If $m < n$, by using a similar argument, we can show that there are $\pi^*(m) + S_2^*$ colors needed to color the graph $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$.

2.8. Theorem: For any variation monogenic semigroup graphs $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$, the clique number of $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$ is given as:

$$\omega(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)) = \begin{cases} \pi^*(n) + S_1^*, & if\ n \leq m \\ \pi^*(m) + S_2^*, & if\ m < n \end{cases}$$

Proof. Suppose that $n < m$. From the definition of $\pi(n)$ and $\pi(m)$, we know that $(x_1^{p_f}, x_2^{p_l}) - (x_1^{p_t} x^{p_h})$ for any $p_f, p_t \in \pi(n)$ and $p_l, p_h \in \pi(m)$. Also, we know from the definition of the tensor product that $(x_1^{p_f}, x_2^{p_l})$ is neither adjacent to $(x_1^{p_t}, x_2^{p_l})$ nor $(x_1^{p_f}, x^{p_h})$. As $\pi^*(n) \leq \pi^*(m)$ shows, there are $\pi^*(n)$ different vertices that are adjacent to each other. Next, consider the set of vertices $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*}), (x_1^{s_{1e}^*}, x_2^{s_{2q}^*})$ such that $s_{1f}^*, s_{1e}^* \notin \pi(n)$, $s_{2l}^*, s_{2q}^* \notin \pi(m)$, $\lceil \frac{n}{2} \rceil < s_{1f}^*, s_{1e}^* \leq n$, and $\lceil \frac{m}{2} \rceil < s_{2l}^*, s_{2q}^* \leq m$. From the definition of S_1^* and S_2^*, we know that $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*}) - (x_1^{s_{1e}^*}, x_2^{s_{2q}^*})$ and $(x_1^{p_f}, x_2^{p_l}) - (x_1^{s_{1f}^*}, x_2^{s_{2l}^*})$ where $p_f \in \pi(n)$ and $p_l \in \pi(m)$. Since $S_1^* \leq S_2^*$, there are more S_1^* vertices in the clique.

Suppose that p_1 and p_2 are the least prime numbers in $\pi(n)$ and $\pi(m)$, respectively. Consider the vertices (x_1^k, x_2^z), where $1 \leq k < p_1$ and $1 \leq z < p_2$. The vertex (x_1^k, x_2^z) is not adjacent to $(x_1^{p_f}, x_2^{p_l})$ since $k + p_1 < n$ and $k + p_2 < m$. Moreover, (x_1^k, x_2^z) is not adjacent to $(x_1^{s_{1f}^*}, x_2^{s_{2l}^*})$ since $k + s_{1f}^* < n$ and $k + s_{2l}^* < m$. Therefore, the clique number is $\pi^*(n) + S_1^*$ when $m < n$, and by using a similar argument, we can show that $\pi^*(m) + S_2^*$ is the clique number.

Using the knowledge of relatively prime numbers to study the structure of tensor product graphs of the variation monogenic semigroup, as well as observing Theorem 2.7 and Theorem 2.8, since

$$\mathcal{X}(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)) = \omega(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)) = \begin{cases} \pi^*(n) + S_1^*, & if\ n \leq m \\ \pi^*(m) + S_2^*, & if\ m < n \end{cases}$$

we discover that the graph $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$ preserves the perfectness property, as stated in the next lemma.

2.9. Lemma: The graph $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$ is a perfect graph.

The tensor product graph of the monogenic semigroup $\Gamma(S_{Mn}^1) \otimes \Gamma(S_{Mm}^2)$ of Akgunes *et al.* (2014) proved that $\mathcal{X}\left(\Gamma(S_{Mn}^1) \otimes \Gamma(S_{Mm}^2)\right) = Min\left(\mathcal{X}\left(\Gamma(S_M^1)\right), \mathcal{X}\left(\Gamma(S_M^2)\right)\right)$. This result is also true for the tensor product graph of the variation monogenic semigroup, which can be deduced from Theorem 2.7. So, $\mathcal{X}\left(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)\right) = Min(\mathcal{X}(\Gamma(VS_M^1)), \mathcal{X}(\Gamma(VS_M^2)))$.

From Akgunes *et al.* (2014), the domination number of the tensor product of the graph of the monogenic semigroup is 3. However, the domination number of the tensor product of the variation monogenic semigroup graph cannot possibly be 3, as shown below and from Theorem 2.11 in Al Subaiei and Akwu (2020).

2.10. Theorem: For any variation monogenic semigroup graphs $\Gamma(VS_{Mn}^1)$ and $\Gamma(VS_{Mm}^2)$, the domination number of $\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)$ is given as:

$$\gamma(\Gamma(VS_{Mn}^1) \otimes \Gamma(VS_{Mm}^2)) = \begin{cases} 2\gamma(\Gamma(VS_{Mn}^1))\gamma(\Gamma(VS_{Mm}^2)), & n\ or\ m\ prime \\ \gamma(\Gamma(VS_{Mn}^1))\gamma(\Gamma(VS_{Mm}^2)), & Otherwise \end{cases}$$

Proof. According to Al Subaiei and Akwu (2020), the domination number of any graph $\Gamma(VS_{Mn})$ is as follows:

- If n is prime, then from $\gamma(\Gamma(VS_{Mn})) = 1$ say x^n.
- If n is not prime and $n - p$ does not contain a prime number t such that $gcd(n,t) \neq 1$, where p is the highest prime number less or equal to n. Then, we need one more vertex, say x^p, such that $\gamma(\Gamma(VS_{Mn})) = 2 = |\{x^n, x^p\}|$.
- If $n - p$ contains a prime number t such that $gcd(n,t) \neq 1$ and $p + t < n$, then we need the additional vertex $x^r, p < r < n$ such that $r + t > n$ and $gcd(r,t) = 1$. Therefore, $\gamma(\Gamma(VS_{Mn})) = 3 = |\{x^n, x^r, x^p\}|$.
- If $n - p$ contains t and q where t is a prime number and $q \equiv 0 \ (mod\ t)$ such that $gcd(n,t) \neq 1$, $p + t < n$ and $p + q < n$. Also, for r, where $p < r < n$, $gcd(r,t) = 1$, and $r + t > n$ but $gcd(r,q) \neq 1$. Also, since $gcd(n,t) \neq 1$, we have $gcd(q,n) \neq 1$. Therefore, we need vertex x^s, $p < s < n$ such that $s + q > n$ and $gcd(s,q) = 1$. Then, $\gamma(\Gamma(VS_{Mn})) = 4 = |\{x^n, x^r, x^s, x^p\}|$.

Now, from the above, let $\gamma(\Gamma(VS^1_{Mn})) = k$ and $\gamma(\Gamma(VS^2_{Mm})) = l$. Next, we split the problem into the following two cases:

Case 1: When n and m are both prime or when n or m is prime.

If n is prime and m is not or n and m are both prime, let D represent the domination number and construct $(k+1) \times l$ Latin square, where $k + 1 = \{x^n, x^{n-1}\}$. Also, if m is prime and n is not, construct $k \times (l+1)$ Latin square, where $l + 1 = \{x^m, x^{m-1}\}$. Then, the Latin square $(k+1) \times l$ or $k \times (l+1)$ gives $2\gamma(\Gamma(VS^1_{Mn}))\gamma(\Gamma(VS^2_{Mm}))$. All vertices of $\Gamma(VS^1_{Mn}) \otimes \Gamma(VS^2_{Mm}) \backslash 2\gamma(\Gamma(VS^1_{Mn}))\gamma(\Gamma(VS^2_{Mm}))$ are adjacent to at least one vertex in $2\gamma(\Gamma(VS^1_{Mn}))\gamma(\Gamma(VS^2_{Mm}))$. Therefore, we have $D = 2\gamma\left(\Gamma(VS^1_{Mn})\right)\gamma\left(\Gamma(VS^2_{Mm})\right)$, which is the desired result.

Case 2: When both n and m are not prime.

Construct Latin square $k \times l$ where k takes care of n number of rows and l takes care of m number of columns. Hence, the Latin square $k \times l$ gives $\gamma\left(\Gamma(VS^1_{Mn})\right)\gamma\left(\Gamma(VS^2_{Mm})\right)$, which is the domination number of $\Gamma(VS^1_{Mn}) \otimes \Gamma(VS^2_{Mm})$ whenever n and m are not prime, since all vertices in $\Gamma(VS^1_{Mn}) \otimes \Gamma(VS^2_{Mm}) \backslash \gamma(\Gamma(VS^1_{Mn}))\gamma(\Gamma(VS^2_{Mm}))$ are adjacent to at least one vertex in $\gamma(\Gamma(VS^1_{Mn}))\gamma(\Gamma(VS^2_{Mm}))$.

2.11. Example: Consider the graph $\Gamma(VS_{M10}) \otimes \Gamma(VS_{M9})$. Then, $\mathcal{X}(\Gamma(VS_{M10}) \otimes \Gamma(VS_{M9})) = 4$, $\gamma(\Gamma(VS_{M10}) \otimes \Gamma(VS_{M9})) = 6$, and $\omega(\Gamma(VS_{M10}) \otimes \Gamma(VS_{M9})) = 4$. While the graph $\Gamma(VS_{M8}) \otimes \Gamma(VS_{M6})$ *has* $\mathcal{X}(\Gamma(VS_{M8}) \otimes \Gamma(VS_{M6})) = 3$, $\gamma(\Gamma(VS_{M8}) \otimes \Gamma(VS_{M6})) = 4$, and $\omega(\Gamma(VS_{M8}) \otimes \Gamma(VS_{M6})) = 3$.

Lastly, we draw our conclusion with the following remark.

2.12. Remark: The tensor product graph $\boldsymbol{\Gamma(VS^1_{Mn}) \otimes \Gamma(VS^2_{Mm})}$ can be viewed as a coprime graph. By Equation $(*)$ and definition of $\boldsymbol{\Gamma(VS^1_{Mn}) \otimes \Gamma(VS^2_{Mm})}$, we have that any pair of vertices in $\boldsymbol{\Gamma(VS^1_{Mn}) \otimes \Gamma(VS^2_{Mm})}$ are adjacent if and only if their labels are relatively prime.

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References

Akgunes, N., Das, K. C. and Sinan Cevik, A. (2014). Some properties on the tensor product of graphs obtained by monogenic semigroups. *Applied Mathematics and Computation,* **235**, 352–7.

Al Subaiei, B. and Akwu, A. D. (2020). The zero-divisor graph of variation monogenic semigroups. *Sci. J. King Faisal Univ.,* **21**(2), 189–93. DOI: 10.37575/b/sci/0016

Anderson, D. F. and Badawi, A. (2017). 'The zero-divisor graph of a commutative semigroup: A survey'. *In*: M. Droste, L. Fuchs, B. Goldsmith and L. Strüngmann. (eds.) *Groups, Modules, and Model Theory - Surveys and Recent Developments,* 23–39. Cham: Springer. DOI: 10.1007/978-3-319-51718-6_2

Asmerom, G. A. (1998). Imbeddings of the tensor product of graphs where the second factor is a complete graph. *Discrete Mathematics,* **182**(n/a), 13–9.

Bondy, J. A. and Murty, U. S. (1976). *Graph Theory with Applications.* New York: Elsevier Science Publishing Co., Inc.

Das, K. C., Akgunes, N. and Sinan Cevik, A. (2013). On a graph of monogenic semigroups. *Journal of Inequalities and Applications,* **2013**(1), 1–13.

DeMeyer, F. R. and DeMeyer, L. (2005). The zero-divisor graphs of semigroups. *J. Algebra,* **283**(1), 190–8.

DeMeyer, F. R., McKenzie, T. and Schneider, K. (2002). The zero-divisor graph of a commutative semigroups. *Semigroup Forum,* **65**(2), 206–14.

Harary, F. and Trauth, Jr., C. A. (1966). Connectedness of products of two directed graphs. *SIAM Journal on Applied Mathematics,* **14**(2), 250–4.

Lovasz, L. (1972). Normal hypergraphs and the weak perfect graph conjecture. *Discrete Math,* **2**(3), 253–67.

Sampathkumar, E. (1972). On tensor product graphs. *Journal of the Australian Mathematical Society,* **20**(3), 268–73.

Wright, S. E. (2007). Lengths of paths and cycles in zero-divisor graphs and digraphs of semigroups. *Commun. Algebra,* **35**(6), 1987–91.